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A dynamic system approach to the characterization of a class of rheological materials at base-temperatures

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Abstract

A powerful dynamic system identification method is proposed for the determination of the characteristic response function of a class of dissipative (e.g., rheological) material-systems from an experimental dynamic system analysis. In this context, a model of rational function of polynomials for the so-called "*transfer response function*" is assumed. In this context, a discrete-time system analysis method is first introduced to identify the order and parameters of the model. Second, the characteristic function of the system is obtained by using an inverse integral method. The numerical scheme and pertaining examples for testing the model are presented. It is concluded that the proposed procedure, although powerful, is easy to use, and the pertaining model is accurate and efficient.

Keywords: Dynamic; System; Model; Viscoelastic; Material; Numerical

1. Introduction

A fundamental task of most physical sciences is the establishment of mathematical models for the analysis, prediction and control of physical processes. Such models may be obtained by adopting, for instance, one of the following two approaches:

1. Physical reasoning by observing the behaviour of the physical process.
2. Mathematical modelling that would be based on the analysis of experimental data concerning the system.

The first approach is based mainly on the analysis of the basic constitutive properties of the system. The second approach, however, does not concern itself about the type of the physical system it is dealing with. Instead, it analyses and establishes a model from the experimental input and output data and/or signals concerning the system.

For a linear dissipative system, it is well recognized that the behavioural functions characterizing both quasi-static and dynamic responses are interrelated. Quasi-static experiments, to determine such response functions, require, however, considerably long periods of time to be performed. To overcome such inconvenience, dynamic methods are recently attracting the attention of researchers. Gibson et al. [1], for instance, presented a method by which experimental dynamic data are used to determine both quasi-static and dynamical response behaviour of the material. In their method, the complex moduli were obtained first from vibration measurements by employing Fast Fourier Transform technique. Then, the quasi-static time-dependent properties were calculated from the experimentally determined dynamic properties by employing a numerical integration algorithm.

A new method is introduced in this paper by considering the dissipative material as a dynamical system. A relation is first established between the quasi-static response functions and corresponding frequency functions of a specifically proposed dynamical system. In the frequency domain, an analytical model is assumed for the frequency response function of the system, then, a discrete-time system analysis is developed to estimate the order and parameters of the proposed model. The efficiency and accuracy of the proposed method are demonstrated through a number of numerical examples.

2. The model

For a linearly dissipative material system, the relationship between the stimulus (input) and the output can be written in the

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following general form [2,3]:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)g(t-\tau)\,\mathrm{d}\tau \tag{1}$$

where in the relaxation phase:

$$x(t) = \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}, \qquad g(t) = R(t), \qquad y(t) = \sigma(t) \tag{2}$$

and in the creep phase:

$$x(t) = \frac{\mathrm{d}\sigma(t)}{\mathrm{d}t}, \qquad g(t) = C(t), \qquad y(t) = \varepsilon(t) \tag{3}$$

In the above two equations, $\varepsilon(t)$ and $\sigma(t)$ are the time-dependent strain and stress, respectively, $R(t)$ the relaxation function and $C(t)$ is the creep function. For simplicity, we will refer to Situation (2), Eq. (2) above, as the "*relaxation experiment*", and to Situation (3) as the "*creep experiment*".

From a system theory point of view, Eq. (1) represents a relationship between an input $x(t)$ and a corresponding output $y(t)$ of the system with $g(t)$ being the "*system characteristic function*".

Therefore, if one considers the dissipative material system as a dynamical system, then, the characterization of its time-dependent behaviour would be a process of identification of the corresponding system from dynamical measurements.

Taking Fourier transform of $y(t)$ and $x(t)$ and denoting:

$$Y(\mathrm{i}\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{4}$$

and

$$X(\mathrm{i}\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{5}$$

Then, by combining Eqs. (1) and (4) it follows that

$$Y(\mathrm{i}\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega\phi} g(\phi)\,\mathrm{d}\phi \int_{-\infty}^{\infty} x(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau \tag{6}$$

where φ is the time parameter $(t–\tau)$ and ω indicates the frequency. Thus, by combining (5) and (6), one has the following relation in frequency domain:

$$Y(\mathrm{i}\omega) = 2\pi H(\mathrm{i}\omega)X(\mathrm{i}\omega) \tag{7}$$

where

$$H(\mathrm{i}\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t} g(t)\,\mathrm{d}t \tag{8}$$

With reference to Eq. (8), $H(\mathrm{i}\omega)$ is the Fourier transform of the system characteristic function $g(t)$. Meantime, the frequency response function of the system is identified as

$$F(\mathrm{i}\omega) = 2\pi H(\mathrm{i}\omega) \tag{9}$$

In terms of the frequency response function (9), Eq. (7) becomes

$$Y(\mathrm{i}\omega) = F(\mathrm{i}\omega)X(\mathrm{i}\omega) \tag{10}$$

Denoting the inverse Fourier transform of the frequency response function $F(\mathrm{i}\omega)$ by $f(t)$, then, in view of expressions (8) and (9), it follows that

$$f(t) = \int_{-\infty}^{\infty} F(\mathrm{i}\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = 2\pi g(t) \tag{11}$$

Thus, Eq. (11) implies that the frequency response function of the dynamic system is the Fourier transform of the characteristic function $g(t)$ of the system multiplied by 2π.

To model the behaviour of the dissipative material system under consideration, we assume [3] that the frequency response function of the corresponding dynamic system has the following form:

$$F(\mathrm{i}\omega) = \frac{a}{(\mathrm{i}\omega)^p + b_1(\mathrm{i}\omega)^{p-1} + \cdots + b_{p-1}(\mathrm{i}\omega) + b_p} \tag{12}$$

where a and $b_1, b_2, \ldots, b_p$ are constant parameters.

In correspondence to Eq. (12) above, the system characteristic function $g(t)$ is derived as follows:

Assuming the following pth-order algebraic equation:

$$s^p + b_1 s^{p-1} + \cdots + b_{p-1}s + b_p = 0 \tag{13}$$

with roots $\xi_1, \xi_2, \ldots, \xi_p$, Eq. (12) can, then, be written as

$$F(\mathrm{i}\omega) = \frac{a}{(\mathrm{i}\omega - \xi_1)(\mathrm{i}\omega - \xi_2)\cdots(\mathrm{i}\omega - \xi_p)} \tag{14}$$

Further, the above equation can be expressed in a partial fraction form as

$$F(\mathrm{i}\omega) = \sum_{m=1}^{p} \frac{A_m}{\mathrm{i}\omega - \xi_m} \quad (m = 1, 2, \ldots, p) \tag{15}$$

where A_m $(m = 1, 2, \ldots, p)$, corresponding to roots ξ_m $(m = 1, 2, \ldots, p)$, are calculated by

$$A_m = \frac{a}{\prod_{j=1}^{p}(\xi_m - \xi_j)} \quad (j, m = 1, 2, \ldots, p, \quad j \neq m) \tag{16}$$

Taking the inverse Fourier transform of Eq. (15), one obtains

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} F(\mathrm{i}\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \sum_{m=1}^{p} \frac{A_m}{\mathrm{i}\omega - \xi_m}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \\ &= \sum_{m=1}^{p} A_m\,\mathrm{e}^{\xi_m t} u(t) \quad (m = 1, 2, \ldots, p) \end{aligned} \tag{17}$$

where $u(t)$ is the Heaviside function. Thus, Eq. (17) can be written as

$$f(t) = \begin{cases} \sum_{m=1}^{p} A_m\,\mathrm{e}^{\xi_m t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad (m = 1, 2, \ldots, p) \tag{18}$$

From Eqs. (2) and (17), the relaxation function $R(t)$, in a dynamic relaxation experiment, is expressed as

$$R(t) = \frac{1}{2\pi} f(t) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{p} A_m\, e^{\xi_m t}, & t \geq 1 \\ 0, & t < 0 \end{cases} \quad (m = 1, 2, \ldots, p) \tag{19}$$

On the other hand, if the experiment is a dynamic creep experiment, then, the system characteristic function $g(t)$ represents the creep function. Thus, the expression for the creep function $C(t)$, corresponding to (19), can be written as

$$C(t) = \frac{1}{2\pi}\sum_{m=1}^{p} A_m\, e^{\xi_m t} u(t) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{p} A_m\, e^{\xi_m t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad (m = 1, 2, \ldots, p) \tag{20}$$

3. Numerical evaluation

To test the accuracy and efficiency of the proposed model, a number of numerical illustrations are carried out. The formalism of these illustrations is outlined as follows:

1. For a given system, calculate the response under certain dynamic loading by a numerical method. Then, two discrete-time series (one is the input into the system and the other is the response) are obtained.
2. Assuming that no other knowledge about the system is given, except the following experimental two discrete-time series, determine, first, the parameters of the discrete-time system function, then, establish the corresponding continuous system function.

$$y_i = y(\Delta Ti), \qquad x_i = x(\Delta Ti) \quad (i = 0, 1, 2, \ldots) \tag{21}$$

As an illustrative example, we consider below the first-order system:

$$\dot{y} + 5y = x(t) \tag{22}$$

under an input represented by: $x(t) = 100 \sin(t^{1.5})$.

From the model above, the parameters of system (22) can be determined as

$$p = 1, \quad b_1 = 5 \text{ and } a = 1.0$$

With an input $x(t) = 100 \sin(t^{1.5})$, which may be the rate of strain or stress, one obtains a discrete-time series of output as plotted, with $\Delta T = 0.01$, in Fig. 1. One uses, then, discrete-time systems (DTS's) of different orders to model the system. The errors pertaining to three different discrete-time systems were calculated and are listed in Table 1.

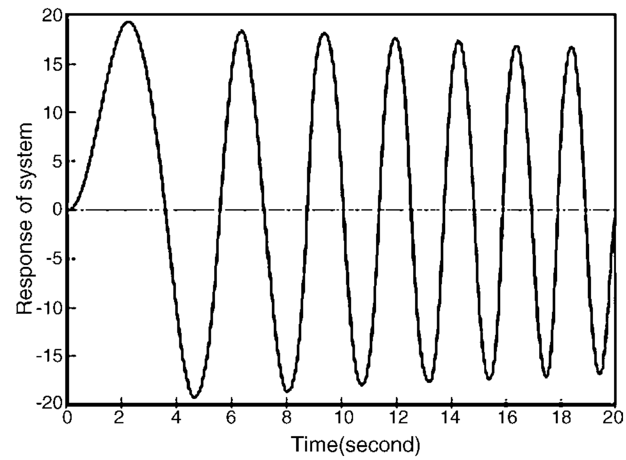

Fig. 1. Output $y(t)$ corresponding to the input: $x(t) = 100 \sin(t^{1.5})$ with $\Delta T = 0.01$. First-order system function $\dot{y} + 5y = x(t)$ with parameters $a = 1.0$, $b_1 = 5$ and an order $p = 1$.

Table 1
Errors in determining three different discrete-time systems

Order	Error
First	0.852409E−02
Second	0.253536E+00
Third	0.634591E−1

From Table 1, above, the DTS of first order is the system with minimum error, therefore, one chooses this first-order DTS to model the continuous system governed by the first-order differential equation of (22).

In this context, Fig. 2 shows the exact and estimated responses given by the first-order DTS. Meanwhile, Fig. 3 shows the exact and estimated values of the system characteristic function $g(t)$ as obtained from the arrived at first-order DTS.

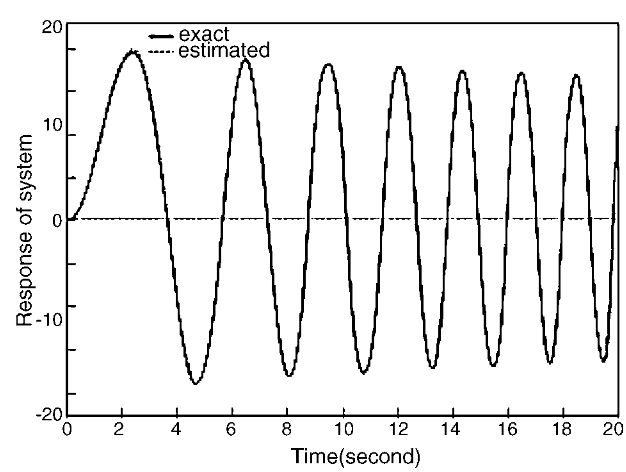

Fig. 2. The exact and the estimated responses from the first-order DTS. First-order system: function $\dot{y} + 5y = x(t)$ with parameters $a = 1.0$, $b_1 = 5$, $p = 1$ and the input $x(t) = 100 \sin(t^{1.5})$ with $\Delta T = 0.01$.

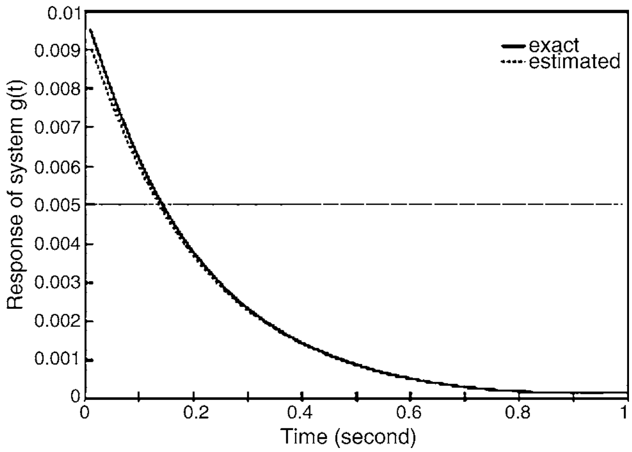

Fig. 3. The exact and the estimated system characteristic values of the characteristic function $g(t)$ as derived from the first-order DTS. First-order continuous time system: function $\dot{y}+5y=x(t)$ with parameters $a=1.0$, $b_1=5$, $p=1$ and the input: $x(t)=100\sin(t^{1.5})$ with $\Delta T=0.01$.

4. Conclusions

A new method for dynamic identification of a class of linear dissipative (e.g., rheological) systems is developed. In the presented method, a rational function of polynomials model for the frequency response function, of an associated dynamic system, is first developed. A discrete-time system analysis method is, then, introduced to identify the order and parameters of the model directly from the input and output signals of the system. Numerical examples show that the proposed model is efficient and accurate.

Acknowledgment

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